Self-Organized Criticality in Models of Complex System Dynamics

Yury M. Pis'mak

Department of Theoretical Physics,
State University Saint-Petersburg

Dual Year Spain-Russia
Particle Physics, Nuclear Physics
and Astroparticle Physics,
Barcelona, Spain, 5-12 November 2011
In the framework of the BSM the evolution of biological ecosystem is described as follows. The state of the ecosystem of $N$ species is characterized by a set $\{x_1, \ldots, x_N\}$ of $N$ number, $0 \leq x_i \leq 1$. The number $x_i$ represents the barrier of the $i$-th species toward further evolution. Initially, each $x_i$ is set to a randomly chosen value. At each time step the barrier $x_i$ with minimal value and the barriers of its "nears neighbors" are replaced by new random numbers. In the random neighbor model (RNM) [5] the $K-1$ of the specie with minimal barriers are chosen at random. The interaction structure in the local model nears neighbors is given a near neighbor graph (for the lattice in D-dimensional space). Thus, for each species in the LM the nearest neighbors are assumed to be fixed.
Summary

For the LM the most of results are obtained by numerical experiments or in the framework of mean field approximation. The RNM is more convenient for analytical studies. The master equations obtained in [6] for RNM are very useful for this aim. These equations appeared to be exact solvable. The stationary solution was found in [6]. The time dependent solutions were obtained in [7] (infinite system), [8] (finite system).
State of system after large time of evolution

Yury M. Pis'mak
SOC in Models of Complex System Dynamics
Master equations for RNM

The master equations for the RNM are obtained in [3]. They are of the form:

\[ P_n(t + 1) = A_n P_n(t) + B_{n+1} P_{n+1}(t) + C_{n-1} P_{n-1}(t) + \]
\[ + D_{n+2} P_{n+2}(t) + (B_1 \delta_{n,0} + A_1 \delta_{n,1} + C_1 \delta_{n,2}) P_0(t). \]
Master equations for RNM

The master equations for the RNM are obtained in [3]. They are of the form:

\[ P_n(t + 1) = A_n P_n(t) + B_{n+1} P_{n+1}(t) + C_{n-1} P_{n-1}(t) + \]
\[ + D_{n+2} P_{n+2}(t) + (B_1 \delta_{n,0} + A_1 \delta_{n,1} + C_1 \delta_{n,2}) P_0(t). \]

Here, \( P_n(t) \) is the probability that \( n \) is the number of barriers having values less than a fixed parameter \( \lambda \) at the time \( t \); \( 0 \leq n \leq N \), \( 0 \leq \lambda \leq 1 \), \( t \geq 0 \). The initial probability distribution \( P_n(0) \) is supposed to be given. For \( 0 < n \leq N \)

\[ A_n = 2\lambda(1 - \lambda) + \frac{n - 1}{N - 1} \lambda(3\lambda - 2), \]
\[ B_n = (1 - \lambda)^2 + \frac{n - 1}{N - 1} (1 - \lambda)(3\lambda - 1), \]
\[ C_n = \lambda^2 - \frac{n - 1}{N - 1} \lambda^2, \quad D_n = (1 - \lambda)^2 \frac{n - 1}{N - 1}. \quad (1) \]

\( A_n = B_n = C_n = D_n = 0 \) for \( n = 0 \) and \( n > N \).
Dynamical variables and parameters

It holds

\[ P_n(t) \geq 0, \sum_{n=0}^{N} P_n(t) = 1. \]

For analysis of master equations it is convenient to introduce the generating function \( q(z, u) \):

\[ q(z, u) \equiv \sum_{t=0}^{\infty} \sum_{n=0}^{N} P_n(t) z^n u^t. \]

In virtue of definition, \( q(z, t) \) is polynomial in \( z \), analytical in \( u \) for \( |u| < 1 \) and

\[ q(1, u) = \frac{1}{1-u}. \]
The master equations can be rewritten for the generating function $q(z, u)$ as follows:

$$\frac{1}{u}(q(z, u) - q(z, 0)) = (1 - \lambda + \lambda z)^2 \left( \frac{1}{z} \left( 1 - \frac{1 - z}{N - 1} \left( \frac{1}{z} - \frac{\partial}{\partial z} \right) \right) \times \right)$$

$$\times (q(z, u) - q(0, u)) + q(0, u).$$

The function $q(z, 0) = \sum_{n=0}^{N} P_n(0)z^n$ in (8) is assumed to be given.
Exact solution of the master equations

The generating function $q(z, u)$ can be presented in the form

$$q(z, u) = z \frac{N - 1}{u} e^{R(z,u)} \int_{z}^{\lambda-1} e^{-R(x,u)} \frac{q(x,0)dx}{(1 - \lambda + \lambda x)^2(1 - x)} +$$

$$+ q(0, u)(1 + z \frac{N - 1}{u} e^{R(z,u)} \int_{z}^{\lambda-1} e^{-R(x,u)} \frac{1 - u(1 - \lambda + \lambda x)^2)dx}{(1 - \lambda + \lambda x)^2(1 - x)}),$$

where

$$q(0, u) = \frac{1}{\frac{\lambda-1}{\lambda}} \int_{\frac{\lambda-1}{\lambda}}^{1} e^{-R(x,u)} \frac{q(x,0)dx}{(1-\lambda+\lambda x)^2(1-x)}.$$

Yury M. Pis'mak, SOC in Models of Complex System Dynamics.
Infinite system with $K$ random neighbors

The master equation in the case $N = \infty$ and $K$ interacting random neighbors is of the form

$$P_n(t + 1) = \sum_{l=0}^{K} C_n^l \theta(K - l) \lambda^l (1 - \lambda)^{K-l} P_{n-l+1}(t) +$$

$$+ \theta(K - n) C_K^n \lambda^n (1 - \lambda)^{K-n} P_0(t).$$

Here, $P_n(t)$ is the probability that $n$ is the number of barriers having values less than a fixed parameter $\lambda$ at the time $t$; $0 \leq \lambda \leq 1$, $t \geq 0$ $n \geq 0$. For compact writing we used the threshold function $\theta(n)$ defined as follows

$\theta(n) = 1$ for $n \geq 0$ and $\theta(n) = 0$ for $n < 0$. The initial probability distribution $P_n(0)$ is supposed to be given.
Master equation for $q(z, u)$

The generating function $q(z, u)$ fulfil the equation

$$(z - u(1 - \lambda + \lambda z)^K)q(z, u) = (z - 1)u(1 - \lambda + \lambda z)^K q(0, u) + zq(z, 0).$$

If $\alpha(u)$ is the analytical in $u$ solution of equation

$$\alpha(u) - u(1 + \lambda(\alpha(u) - 1))^K = 0.$$

and

$$\beta(z) = \frac{z}{(1 - \lambda + \lambda z)^K},$$

then

$$\alpha(\beta(z)) = z, \quad \beta(\alpha(u)) = u,$$

and

$$d(y, u) = \frac{q(\alpha(y), u)}{1 - \alpha(y)}$$

is analytical in $y$ and $u$ in the neighborhood of $y = 0$ and $u = 0$:

$$d(y, u) = \sum_{n,t=0}^{\infty} C_n(t)y^n u^t.$$
The coefficients $C_n(t)$ fulfill the equations

$$C_n(t + 1) = C_{n+1}(t) \text{ for } n \geq 0, \ t \geq 0.$$ 

The initial conditions $c_0 = C_n(0)$ are defined by the function $q(z, 0)$.

$$c_n = \frac{1}{2\pi i} \oint_{|z|=\epsilon} \frac{q(z, 0)\beta(z)^{-n}}{(1-z)} \frac{\partial}{\partial z} \ln \beta(z) dz$$

(2)

The equation for $C_n(t)$ has the simple solution:

$$C_n(t) = c_{n+t}.$$
Asymptotic of $c_n$ for large $n$

$$c_n = \sqrt{\frac{K-1}{2\pi Kn}} \left\{ 1 - \frac{K}{2n(K-1)} \left[ M_2 + \frac{K^2 - K + 1}{6K^2} \right] + O \left( \frac{1}{n^2} \right) \right\}$$

for $\lambda = 1/K \equiv \lambda_{cr}$,

$$c_n = \theta(\lambda_{cr} - \lambda)(1 - K\lambda) + \frac{z_0 \sqrt{K} e^{-n\gamma(z_0)}}{\sqrt{2\pi(K-1)n^3}} \left\{ \phi_0 - \frac{z_0^2 K}{2n(K-1)} [\phi_2 + \frac{2(K+1)}{z_0 K} \phi_1 + \frac{K^2 + 11K + 1}{6z_0^2 K^2} \phi_0] + O \left( \frac{1}{n^2} \right) \right\}$$

if $\lambda \neq \lambda_{cr}$. Here, we have used the following notations:

$$M_2 \equiv \sum_{n=0}^{\infty} P_n(0)n^2, \quad \phi_n = \frac{\partial^{n+1}}{\partial z^{n+1}} \left( \frac{q(z,0)}{1-z} \right) \bigg|_{z=z_0}.$$

We have denoted $\gamma(z) \equiv \ln(\beta(z))$. In the saddle point $z_0$ we have $\gamma'(z_0) = 0$. 
Self-Organized Criticality in Software Evolution

The better sources of information about changes in computer programs are version control systems. One of them is Concurrent Versions System (CVS). It keeps information about changes happened in short time intervals.

Using the CVS in our work, we studied the histories of three software projects: Mozilla web browser, Free-BSD Operating System and Gnu Emacs text editor. For each of these projects we analyzed only files written in the basic for the project language. They are: C++ for Mozilla, C for Free-BSD and Lisp for Emacs. Header files for C/C++ were not studied. Total amounts of the processed files are approximately in order 9000, 11000, 900 for Mozilla, Free-BSD, Emacs. Total lengths of RCS-files are $1 \cdot 10^7$, $1 \cdot 10^7$ and $2 \cdot 10^6$ lines. Total amount of the processed data exceeds 2 Gigabytes. Due to some resource limitations only part of the Free-BSD CVS storage ware processed. Histories of all three projects are stored under control of the CVS and were publicly available during our research period from the corresponding Internet servers.

$\log P(A)$ vs $\log A$ graph.
Self-Organized Criticality in Software Evolution. Distribution $P(A)$ for Free-BSD.
Self-Organized Criticality in Software Evolution.
Distribution $P(D)$ for Free-BSD.
Distribution $P(S)$ of temporal durations of avalanches for a one dimensional lattice model
Distribution $P(S)$ of temporal durations of avalanches for a random neighbor model
In our work we studied two modifications of the BS-model: on one-dimensional lattice with periodic boundary conditions and with one random neighbor. The $\alpha$ coefficient always taken as $\frac{1}{2}$. The initial size of the system was 8000 elements. The experiment went on until one million of avalanches registered. For the one-dimensional lattice with periodic boundary conditions version of model we got the following results:

1. $P(S)$ distribution of temporal durations of avalanches. $P(S)$ in log-log scale, is a power law with exponent $\tau = -1.358 \pm 0.005$

2. Distribution $P(A)$ of number of inserted nodes is a power law with exponent $\mu_a = 1.45 \pm 0.01$

3. Distribution $P(D)$ of number of deleted nodes is a power law with exponent $\mu_d = -1.47 \pm 0.02$
For the one random neighbor version of model we got the following results:

1. $P(S)$ distribution of temporal durations of avalanches. At fig. ?? $P(S)$ is shown in log-log scale. $P(S)$ is a power law with exponent $\tau = -1.901 \pm 0.008$

2. Distribution $P(A)$ of number of inserted nodes is a power law with exponent $\mu_a = -1.98 \pm 0.01$

3. Distribution $P(D)$ of number of deleted nodes is a power law with exponent $\mu_d = -2.10 \pm 0.02$
Self-organized criticality seems to be fruitful approach for investigations of evolution of natural complex systems.
Thank you for your attention!