1. In the early universe, the equations of motion of a scalar field are:

\[ \partial_t^2 \phi + 3H \partial_t \phi + V' = 0, \quad H^2 = \frac{8\pi}{3m_{Pl}^2} \left( V + \frac{1}{2} (\partial_t \phi)^2 \right), \]

where \( V(\phi) \) is the potential. Discuss the conditions for “vacuum domination” and “slow roll”. For the potential \( V(\phi) = \frac{1}{2} m^2 \phi^2 \) use the slow roll approximation to obtain the inflationary solutions,

\[ \phi(t) = \phi_i - \frac{m m_{Pl}}{2\sqrt{3\pi}} t, \quad a(t) = a_0 \exp \left[ \frac{2\pi}{m_{Pl}^2} \right] \left[ \phi_i^2 - \phi^2(t) \right], \]

where \( \phi_i \) is the value of the field at the start of inflation, and \( m_{Pl} \) is the Planck mass.

What is the value of \( \phi \) when inflation ends? Find an expression for the number of \( e \)-folds. If \( V(\phi_i) \sim m_{Pl}^4 \) estimate the total number of \( e \)-folds of inflation.
2. In this question, use natural units, $8\pi G = M_{\text{Pl}}^{-2} = 1$, where $M_{\text{Pl}}$ is the reduced Planck mass.

The equations of motion for a flat Robertson-Walker cosmology containing a homogeneous scalar field, $\phi(t)$, with potential energy density $V(\phi)$ are

$$\partial_t^2 \phi + 3H \partial_t \phi + V' = 0, \quad H^2 = \frac{1}{3} \left( V + \frac{1}{2} (\partial_t \phi)^2 \right),$$

where $V' \equiv dV/d\phi$, $\partial_t$ denote derivatives with respect to proper time, and $H$ is the Hubble parameter.

(a) Show that $2\partial_t H = -(\partial_t \phi)^2$.

(b) The power spectrum of these fluctuations is

$$P_R(k) = \left( \frac{H^2}{2\pi \partial_t \phi} \right)^2,$$

where the right hand side is evaluated at Hubble exit, $k = aH$. Using the slow roll approximation, $(\partial_t \phi)^2 \ll V$ and $3H \partial_t \phi \simeq -V'$, show that

$$P_R(k) \simeq \frac{8}{3\epsilon_V} \left( \frac{V^{1/4}}{\sqrt{8\pi}} \right)^4,$$

where $\epsilon_V \equiv \frac{1}{2} \left( \frac{V'}{V} \right)^2$ is the first slow roll parameter.

(c) By showing that $d\ln k/dt \simeq H(1 - \epsilon_V)$, or otherwise, show to leading order in slow roll that the spectral index, defined by $n_s(k) \equiv 1 + d\ln P_R(k)/d\ln k$ is

$$n_s(k) \simeq 1 - 6\epsilon_V + 2\eta_V,$$

where $\eta_V \equiv \frac{V''}{V}$ is the second slow roll parameter.

(d) Show further that

$$dn_s/d\ln k \simeq 16\epsilon_V \eta_V - 24\xi_V^2 - 2\xi_V^2,$$

where $\xi_V^2 \equiv M_{\text{Pl}}^{-4} \frac{V' V''}{V^2}$.